УДК 517.91

NONLOCAL BOUNDARY VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH MAXIMA

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Abstract. In this report it is considered the questions of one value solvability of nonlocal boundary value problems for nonlinear impulsive differential equations with mixed maxima. The problem is reduced to the nonlinear functional integral equations. The method of successive approximations is applied combining it with the method of compressing mapping.

Keywords: impulsive differential equations, one value solvability, method of successive approximations, integral condition.

In this paper we will talk about the features of solving differential equations with maxima and about the unique solvability of a nonlocal boundary value problem for an impulsive system of differential equations with maxima.

1. Monotone solutions of differential equations with maxima

We consider on the interval $[0,T]$ the functional-differential equations of the following form

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau) \middle| \tau \in \left[h_1(t); h_2(t)\right]\right\}\right),$$

where $h_1(t) < h_2(t),\ t \in [0;T].$ This type functional-differential equations are called as differential equations with maxima. We tell how to find monotone solutions of this equation with maxima. The set of increasing solutions of these differential equation with maxima coincides with the set of increasing solutions of the following differential equation with deviation

$$x'(t) = f\left(t, x(t), x\left[h_2(t)\right]\right),\ t \in [0,T].$$

But the decreasing solutions of this differential equation does not satisfy the differential equation with maxima.

The set of decreasing solutions of differential equation with maxima coincides with the set of decreasing solutions of the following differential equation with deviation

$$x'(t) = f\left(t, x(t), x\left[h_1(t)\right]\right),\ t \in [0,T].$$

But the increasing solutions of this differential equation does not satisfy the differential equation with maxima. These facts are important in studying oscillation properties of differential equations with maxima.

The following type differential equations

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau)\middle|\tau \in \left[h_1(t):|:h_2(t)\right]\right\}\right),\ t \in [0,T], \tag{1}$$

where $\left[h_1(t):|:h_2(t)\right] = \left[\min\limits_t\{h_1(t);h_2(t)\};\max\limits_t\{h_1(t);h_2(t)\}\right]$, we call as a differential equation with mixed maxima. We suppose that there exist some points $t_i \in (0,T),\ i = 1,2,...,p$, at which $h_1(t_i) = h_2(t_i)$. Then on the interval

$$\Omega_1^p = [0,t_1] \cup [t_2,t_3] \cup [t_4,t_5] \cup ... \cup [t_{p-1},t_p]$$

the differential equation with mixed maxima (1) has the form

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau)\middle|\tau \in \left[h_1(t);h_2(t)\right]\right\}\right). \tag{2}$$

On the interval

$$\Omega_2^p = [t_1,t_2] \cup [t_3,t_4] \cup [t_5,t_6] \cup ... \cup [t_p,T]$$

the differential equation with mixed maxima (1) has the form

$$x'(t) = f\left(t, x(t), \max\left\{x(\tau)\middle|\tau \in \left[h_2(t);h_1(t)\right]\right\}\right). \tag{3}$$

The set of solutions of the differential equation with mixed maxima (1) on the interval $[0,T]$ coincides with the union of sets of the solutions of two differential equations (2) and (3) on the intervals Ω_1^p and Ω_2^p, respectively. At the points $t_1,t_2,t_3,...,t_{p-1},t_p$ the solutions of differential equation (1) with mixed maxima have discontinuities depending from the posed problem for differential equations (2) and (3) with deviations.

Example 1. On the interval $[0,\infty)$ we consider the following differential equation with mixed maxima

$$x'(t) = 2\max\left\{x(\tau)\middle|\tau \in \left[t:|:\sqrt{t}\right]\right\},\ t \in [0,\infty). \tag{4}$$

On the interval $[0,1]$ the differential equation (4) with mixed maxima has the form

$$x'(t) = 2\max\left\{x(\tau)\middle|\tau \in \left[t;\sqrt{t}\right]\right\},\ t \in [0,1]. \tag{5}$$

On the interval $[1,\infty)$ the differential equation (4) with mixed maxima has the following form

$$x'(t) = 2\max\left\{x(\tau)\middle|\tau \in \left[\sqrt{t};t\right]\right\},\ t \in [1,\infty). \tag{6}$$

Therefore, solutions of the differential equation (4) with mixed maxima on the interval $[0,\infty)$ have the form

$$x(t) = \begin{cases} \begin{cases} A\cdot t^2,\ t\in[0,1],\ \ A>0, \\ A\cdot e^{2t},\ t\in[0,1],\ \ A<0; \end{cases} \\ \begin{cases} B\cdot t^2,\ t\in[1,\infty)\ \ B<0, \\ B\cdot e^{2t},\ t\in[1,\infty),\ \ B>0. \end{cases} \end{cases}$$

In finding these solutions we solved the differential equations (5) and (6) with maxima.

If we do not specify a continuous gluing condition at a point $t=1$, then naturally, the solution of a differential equation (4) with mixed maxima suffers a discontinuity of the first kind at this point. For example, if we solve the differential equation (4) with mixed maxima with condition $x(0)=-2$ on the first interval $[0,1]$ and solve the differential equation (4) with mixed maxima with condition $x(1)=2$ on the second interval $[1,\infty)$, then we have corresponding solutions $x(t)=-2e^{2t}$ on $[0,1]$ and $x(t)=2e^{2(t-1)}$ on $[1,\infty)$. So, from these solutions we have

$$\lim_{t\to 1-0} x(t) = \lim_{t\to 1-0}\left(-2e^{2t}\right) = -2e^2,\ \ \lim_{t\to 1+0} x(t) = \lim_{t\to 1+0}\left(2e^{2(t-1)}\right) = 2.$$

Consequently, for the difference of limit values of these solutions we obtained discontinuity

$$\lim_{t\to 1+0} x(t) - \lim_{t\to 1-0} x(t) = 2+2e^2.$$

Example 2. On the interval $[0,\infty)$ we consider the following differential equation with mixed maxima

$$x'(t) = \frac{e^t}{(e^t+1)^2}\cdot\frac{e^{\left(1+(-1)^{[t]}\right)t}+1}{e^{\left(1+(-1)^{[t]}\right)t}}\max\left\{x(\tau)\middle|\tau\in\left[t:|:\left(1+(-1)^{[t]}\right)t\right]\right\}, \tag{7}$$

where $[t]$ is the integer part of t.

On the interval $\Omega_1=[0,1]\cup[2,3]\cup[4,5]\cup\ldots$ the differential equation (7) with mixed maxima has the form

$$x'(t) = \frac{2e^t}{(e^t+1)^2}\max\left\{x(\tau)\middle|\tau\in[0;t]\right\}. \tag{8}$$

On the interval $\Omega_2 = [1,2] \cup [3,4] \cup [5,6] \cup ...$ the differential equation (7) with mixed maxima has the following form

$$x'(t) = \frac{e^{2t}+1}{e^t(e^t+1)^2} \max\left\{x(\tau) \middle| \tau \in [t;2t]\right\}. \quad (9)$$

Therefore, solutions of the differential equation (7) with mixed maxima on the interval $[0,\infty)$ have the form

$$x(t) = \begin{cases} \begin{cases} x(t) = C_i \cdot e^{-2(e^t+1)^{-1}}, \ t \in \Omega_1, \ C_i > 0, \\ x(t) = C_j \cdot \dfrac{e^t}{e^t+1}, \ t \in \Omega_1, \ C_j < 0; \end{cases} \\ \begin{cases} x(t) = D_i \cdot \dfrac{e^t}{e^t+1}, \ t \in \Omega_2, \ D_i > 0, \\ x(t) = D_j \left(1+e^{-t}\right) \cdot e^{-2e^{-t}(e^t+1)^{-1}}, \ t \in \Omega_2, \ D_j < 0. \end{cases} \end{cases}$$

In finding these solutions we solved the differential equations (8) and (9) with maxima.

It is required to set conditions at each of points $t_k = t_1, t_2, t_3, ..., t_n, ...$ If we do not specify the continuous gluing conditions at these points, the solution of a differential equation (7) with mixed maxima suffers a discontinuity of the first kind at these points.

2. Nonlocal inverse boundary value problem

Impulsive differential equations have important role in the developing of applied sciences [1-7]. So, on the interval $[0,T]$ for the $t \neq t_i$, $i = 1,2,...,p$ we consider the following system of ordinary differential equations

$$x'(t) = Ax(t) + f\left(t, x(t), \max\left\{x(\tau) \mid \tau \in \left[h_1(t,x(t)) :|: h_2(t,x(t))\right]\right\}\right) \quad (10)$$

with nonlinear boundary value condition

$$B(t)x(0) + \int_0^T K(t,s)\,x(s)\,ds = C + \varphi\left(t, \int_0^T H(t,s,x(s))\,ds\right), \quad (11)$$

impulsive effect

$$x\left(t_i^+\right) - e^{-At_i} x\left(t_i^-\right) = e^{-At_i} F_i\left(x\left(t_i\right)\right), \qquad i = 1,2,...,p \quad (12)$$

and additional condition

$$x(\bar{t}) = D \in R^n, \quad D = const, \ \bar{t} \in (0,T), \ \bar{t} \neq t_i, \ i = 1,2,...,p, \tag{13}$$

where $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $A, B(t) \in R^{n\times n}$, $K(t,s) \in R^{n\times n}$ are given matrix and $\det E(t) \neq 0$, $E(t) = B(t) + \int_0^T K(t,s)ds$, $f : [0,T] \times R^n \times R^n \to R^n$, $\varphi : [0,T] \times R^n \to R^n$, $F_i : R^n \to R^n$ are given functions; $C \in R^n$, $0 < h_1 < h_2 < t$, $h_j = h_j(t, x(t))$, $j = 1,2$, $x\left(t_i^+\right) = \lim_{\eta\to 0^+} x\left(t_i + \eta\right)$, $x\left(t_i^-\right) = \lim_{\eta\to 0^-} x\left(t_i - \eta\right)$ are right-sided and left-sided limits of function $x(t)$ at the point $t = t_i$, respectively.

Formulation of problem. To find a pair of quantities $\left\{x(t) \in PC\left([0,T], R^n\right), \ C \in R^n\right\}$, which of first is continuous function for all $t \in [0,T], t \neq t_i, \ i = 1,2,...,p$ satisfying differential equation (10), nonlocal integral condition (11) and for $t = t_i \ i = 1,2,...,p, \ 0 < t_1 < t_2 < ... < t_p < T$ satisfies the limit condition (12) and additional condition (13).

$$x(t) = J(t;x) \equiv \psi(t) + \sum_{0<t_i<T} W(t,t_i) F_i\left(x\left(t_i\right)\right) +$$

$$+E^{-1}(t)\left[\varphi\left(t, \int_0^T H(t,s,x(s))\,ds\right) - \varphi\left(\bar{t}, \int_0^T H(\bar{t},s,x(s))\,ds\right)\right] +$$

$$+\int_0^T W(t,s) f\left(s, x(s), \max\left\{x(\tau) \mid \tau \in \left[h_1(s,x(s)) :|: h_2(s,x(s))\right]\right\}\right) ds \tag{14}$$

for $t \in \left(t_i, t_{i+1}\right]$, $i = 0,1,...,p$, where $\psi(t) = E^{-1}(t) \cdot D \cdot E(\bar{t})$,

$$W(t,s) = \left[G(t,s) - E^{-1}(t)\bar{G}(s)\right] e^{A(T-s)}, \ E(t) = B(t) + \int_0^T K(t,s)ds,$$

$$G(t,s) = \begin{cases} E^{-1}(t) e^{A(t-s)} \left(B(t) + \int_0^t K(t,\theta)d\theta\right), & 0 \le s \le t, \\ -E^{-1}(t) \int_s^T K(t,\theta)\,d\theta\, e^{A(T-s)}, & t < s \le T, \end{cases}$$

$$\overline{G}(s)=\begin{cases} e^{A(\overline{t}-s)}\left(B(\overline{t})+\int\limits_0^{\overline{t}}K(\overline{t},\theta)d\theta\right), & 0\le s\le \overline{t},\\ -\int\limits_s^T K(\overline{t},\theta)\,d\theta\, e^{A(T-s)}, & \overline{t}<s\le T.\end{cases}$$

Theorem. Suppose the following conditions are fulfilled:

(Y1). The constant matrix A such that there holds estimate

$$\| e^{At} \| \le e^{-at},\ 0<a=const,\ t\in[0,T];$$

(Y2). For all $t,s\in[0,T]$ holds

$$\| E^{-1}(t) \| \cdot \left\| \varphi\left(t,\int\limits_0^T H(t,s,\psi(s))\,ds\right)\right\|_{PC} \le M_\varphi<\infty;$$

(Y3). $\| f\left(t,\psi(t),\psi(t)\right)\|_{PC}\le M_f<\infty,\ \max\limits_{i\in\{1,2,\dots,p\}}\| F_i\left(\psi(t_i)\right)\| \le M_F<\infty;$

(Y4). For all $t\in[0,T],\ x,y\in R^n$ holds

$$| f(t,x_1,y_1)-f(t,x_2,y_2)| \le L_1(t)| x_1-x_2 | + L_2(t)| y_1-y_2 |;$$

(Y5). For all $t\in[0,T],\ x\in R^n$ holds

$$|\varphi(t,x_1)-\varphi(t,x_2)|\le L_3|x_1-x_2|;$$

(Y6). For all $t,s\in[0,T],\ x\in R^n$ holds

$$|H(t,s,x_1)-H(t,s,x_2)|\le L_4(s)|x_1-x_2|,\ 0<\int\limits_0^T L_4(s)\,ds<\infty;$$

(Y7). For all $t\in[0,T],\ x\in R^n$ holds

$$|h_j(t,x_1)-h_j(t,x_2)|\le L_{5j}(t)|x_1-x_2|,\ \ j=1,2;$$

(Y8). For all $x,y\in R^n,\ i=0,1,\dots,p$ holds

$$|F_i(x_1)-F_i(x_2)|\le L_{6i}\,|x_1-x_2|;$$

(Y9). $\rho<1,$ where

$$\rho=S_1+2L_3\| E^{-1}(t)\|^2\int\limits_0^T L_4(s)ds+S_2,\ \ S_1=\sum_{i=1}^p L_{6i}\max_{t\in[0,T]}|W(t,t_i)|,$$

$$S_2 = \int_0^T \max_{t\in[0,T]} |W(t,s)| \cdot \Big[L_1(s) + L_2(s)\Big(1 + M_f\,\big(L_{51}(s) + L_{52}(s)\big)\Big)\Big] d\, s.$$

Then the nonlocal inverse boundary value problem (10)-(13) has a unique pair of solution $\left\{ x(t) \in PC\left(\left[0,T\right],R^n\right); C \in R^n \right\}$. The solution $x(t) \in PC\left(\left[0,T\right],R^n\right)$ can be founded from the following iterative process:

$$\begin{cases} x^k(t) = J\left(t; x^{k-1}\right), \ k = 1,2,3,... \\ x^0(t) = \psi(t), \ t \in (t_i, t_{i+1}), \ i = 0,1,2,..., p. \end{cases}$$

The solution we denote by $\omega(t)$, i.e., $x(t) = \omega(t) \in PC\left([0,T],R^n\right)$. Then to find vector C this solution $\omega(t)$ of the equation (14) we substitute into following presentation:

$$C = D \cdot E(\overline{t}) - \varphi\left(\overline{t}, \int_0^T H(\overline{t}, s, x(s))\, ds\right) - \sum_{0<t_i<T} \overline{G}(t_i) e^{A(T-t_i)} F_i\left(x(t_i)\right) -$$

$$-\int_0^T \overline{G}(s) e^{A(T-s)} f\left(s, x(s), \max\left\{x(\tau) \mid \tau \in \left[h_1(s,x(s)) :|: h_2(s,x(s))\right]\right\}\right) ds$$

for $t \in \left(t_i, t_{i+1}\right], \ i = 0,1,..., p.$

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МАКСИМУМДАРЫ БАР ИМПУЛЬСТІК ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕРГЕ АРНАЛҒАН ЛОКАЛДЫ ЕМЕС ШЕТТІК ЕСЕПТЕР

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Аңдатпа. Осы баяндамада аралас максимумдары бар сызықтық емес импульстік дифференциалдық теңдеулер үшін локалды емес шеттік есептердің бірмәнді шешілу мәселелері қаралады. Есеп сызықтық емес функционалды интегралдық теңдеулерге келтіріледі. Компрессиялық бейнелеу әдісімен біріктірілген біртіндеп жуықтау әдісі қолданылады.

Түйін сөздер: импульстік дифференциалдық теңдеулер, бірмәнді шешілімділік, біртіндеп жуықтау әдісі, интегралдық шарт.

НЕЛОКАЛЬНЫЕ КРАЕВЫЕ ЗАДАЧИ ДЛЯ ИМПУЛЬСНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С МАКСИМУМАМИ

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Аннотация. В настоящем докладе рассматриваются вопросы однозначной разрешимости нелокальных краевых задач для нелинейных импульсных дифференциальных уравнений со смешанными максимумами. Задача сводится к нелинейным функциональным интегральным уравнениям. Применяется метод последовательных приближений, сочетающийся с методом компрессионного отображения.

Ключевые слова: импульсные дифференциальные уравнения, однозначная разрешимость, метод последовательных приближений, интегральное условие.