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# ON NON-INCREASING REARRANGEMENTS OF GENERALIZED FRACTIONAL MAXIMAL FUNCTIONS

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**Annotation.** In this paper we consider the generalized fractional-maximal functions. We give an estimates of non-increasing rearrangements of generalized fractional maximal functions. The study of various properties of operators using a generalized fractional-maximal function is sometimes easier than the study of such operators using a generalized potential.

**Keywords:** non-increasing rearrangement, generalized fractional-maximal function, generalized Riesz potentials

Let $L_0 = L_0(R^n)$ be the set of all Lebesgue measurable functions $f : R^n \to C$ ; $\overset{\square}{L}_0 = \overset{\square}{L}_0(R^n)$ is the set of functions $f \in L_0$, for which the non-increasing rearrangement of the $f^*$ is not identical to infinity. Non-increasing rearrangement $f^*$ defined by the equality:

$$f^*(t) = \inf\left\{y \in [0,\infty) : \lambda_f(y) \le t\right\}, \quad t \in R_+ = (0,\infty),$$

where

$$\lambda_f(y) = \mu_n\left\{x \in R^n : \left|f(x)\right| > y\right\}, \quad y \in [0,\infty)$$

is the Lebesgue distribution function [1].

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau; \quad t \in R_+$$

Throughout this work, we will denote by C positive constants, generally speaking, different in different places.

We define the following classes of function.

*Definition 1.* Let $n \in N$, $R \in (0,\infty]$ and $\Phi : (0,R) \to R_+$. We say that the function $\Phi$ belongs to the class $A_n(R)$ if: $\Phi$ decreases and is continuous on $(0,R)$, the function $\Phi(r)r^n \uparrow$ is quasi-increasing on (0,R);

the function $\Phi$ belongs to the class $B_n(R)$ if $\Phi$ decreases and is continuous on $(0,R)$; there exists a constant $C \in R_+$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \le C\Phi(r)r^n,\ r \in (0,R);$$

the function $\Phi$ belongs to the class $E_n(R)$ if

$$\int_0^{r^n} \frac{ds}{\Phi(s^{1/n})s} \le \frac{C}{\Phi(r)},\ \forall t \in (0,R).$$

For example, $\Phi(t) = t^{-\alpha} \in A_n(\infty),\ 0 < \alpha < n$.

$$\Phi(\rho) = \rho^{\alpha-n} \in B_n(\infty)\,(0 < \alpha < n);\quad \Phi(\rho) = ln\frac{eR}{\rho} \in B_n(R),\ R \in R_+.$$

Note that $\Phi \in E_n(R)$ is equivalent to the inequality:

$$\int_0^r \frac{dt}{\Phi(t)t} \le \frac{C}{\Phi(t)}$$

**Lemma 1.** $E_n(R) \subset B_n(R) \subset A_n(R)$.

**Definition 4.** Let $\Phi \in A_n(\infty)$. The *generalized fractional-maximal function* $M_\Phi f$ is defined for the function $f \in L_1^{loc}(R^n)$ by the equality

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is a ball with the center at the point $x$ and radius $r$.

That is, consider the operator $M_\Phi: L_1^{loc}(R^n) \to L_0(R^n)$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0,n)$ we obtain the classical fractional-maximal function $M_\alpha f$:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)|dy.$$

Other types of generalized fractional-maximal functions were considered in [2], [3].

In the works [4]-[6] the generalized Riesz potential was considered using the convolution operator

$$Af(x) = (G * f)(x) = 2\pi^{-n/2} \int_{R^n} G(x-y)f(y)dy;\ f \in E_1(R^n)$$

where the kernel $G(x)$ satisfies the conditions:

$G(x) \cong \Phi(|x|),\ x \in R^n,$ $\quad \int_0^r \Phi(\rho)\rho^{n-1}d\rho \le c\Phi(r)r^n, r \in R_+.$

In the case $G(x) = |x|^{\alpha-n},\ \ \alpha \in [0,n)$ we have the classical Riesz potential $I_\alpha f$:

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

**Theorem 1.** Let $\Phi \in B_n(\infty)$. Then

$$M_\Phi f(x) \lesssim (G * |f|)(x), x \in R^n$$

for every $f \in L_1^{loc}(R^n)$.

**Theorem 2.** Let $\Phi \in A_n(\infty)$. Then there exist a positive constant $C$ depending from $n$ such that

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s),\ t \in (0,\infty)$$

for every $f \in L_1^{loc}(R^n)$.

**Theorem 3.** Let $\Phi \in A_n(\infty)$. Inequality (10) is sharp in the sense that for every $\varphi \in L_0^+(0,\infty;\downarrow)$ there exists a function $f \in L_1(R^n)$ such that $f^* = \varphi$ a.e. on $(0,\infty)$ and

$$(M_\Phi f)^*(t) \ge C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s),\ t \in (0,\infty),$$

where, $C$ is a positive constant which depends only on $n$.

**Theorem 4.** Let $\Phi \in B_n(\infty)$. Then there exists a positive constant $C$ depending from $n$ such that

$$(M_\Phi f)^{**}(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s),\ t \in (0,\infty) \quad (1)$$

for every $f \in L_1^{loc}(R^n)$.

**Remark 1.** It is known [7] that for generalized Riesz potential satisfies the O'Neill inequality

$$(G * f)^{**}(t) \le C_0 \left(\frac{1}{t}\int_0^t G^*(s)ds\right)\int_0^t f^*(\tau)d\tau + \int_t^\infty G^*(\tau)f^*(\tau)d\tau$$

By Theorem 1 it follows that

$$(M_\Phi f)^{**}(t) \le C_0 \left(\frac{1}{t}\int_0^t G^*(s)ds\right)\int_0^t f^*(\tau)d\tau + \int_t^\infty G^*(\tau)f^*(\tau)d\tau \qquad (2)$$

Note that inequality (1) is more precise than inequality (2).

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# ЖАЛПЫЛАНҒАН БӨЛШЕКТІ МАКСИМАЛДЫ ФУНКЦИЯ ЖӘНЕ ОНЫҢ ӨСПЕЙТІН АУЫСТЫРУЫН БАҒАЛАУ

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**Аңдатпа.** Бұл жұмыста біз жалпыланған бөлшекті максималды функцияларды қарастырамыз. Біз жалпыланған бөлшекті максималды функциялардың өспейтін ауыстыруларының бағаларын береміз. Жалпыланған бөлшекті-максималды функцияны қолданатын операторлардың әр түрлі қасиеттерін зерттеу кейде осындай операторларды жалпыланған потенциалмен зерттеуден оңайырақ.

**Түйін сөздер**: : өспейтін ауыстыру, бөлшекті максималды функция.

# ОБОБЩЕННАЯ ДРОБНО-МАКСИМАЛЬНАЯ ФУНКЦИЯ И ОЦЕНКА ЕЕ НЕВОЗРАСТАЮЩЕЙ ПЕРЕСТАНОВКИ

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**Аннотация.** В данной работе мы рассматриваем обобщенные дробно-максимальные функции. Мы даем некоторые оценки невозрастающих перестановок обобщенных дробных максимальных функций. Изучение различных свойств операторов, использующих обобщенную дробно-максимальную функцию, иногда проще, чем изучение таких операторов, использующих обобщенный потенциал.

**Ключевые слова:** невозрастающая перестановка, обобщенная дробно-максимальная функция .