UDC 517.956
IRSTI 27.31.17

ON THE EXISTENCE AND STABILITY OF UNPREDICTABLE OSCILLATIONS FOR A NONLINEAR DIFFERENTIAL EQUATION WITH PIECEWISE ALTERNATELY RETARDED AND ADVANCED ARGUMENT OF GENERALIZED TYPE

D. ARUĞASLAN ÇINÇIN[1], M. TLEUBERGENOVA[2,3], Z. NUGAYEVA[2,3], M. AKHMET[4]

[1] *Süleyman Demirel University, Isparta, Turkey*

[2] *K.Zhubanov Aktobe Regional University, Aktobe, Kazakhstan*

[3] *Institute of Information and Computational Technologies, Almaty, Kazakhstan*

[4] *Middle East Technical University, Ankara, Turkey*

E-mail: duyguarugaslan@sdu.edu.tr; madina_1970@mail.ru; zahira2009.85@mail.ru; marat@metu.edu.tr

Annotation. Recent studies have shown that existence of an unpredictable oscillation for a differential equation amounts to the presence of Poincaré chaos [1,2]. Stimulated by this fact, it is aimed in this study to analyze unpredictable oscillations for a nonlinear differential equation with generalized type piecewise alternately retarded and advanced argument. Existence-uniqueness and exponential stability of the unpredictable oscillations are verified for the considered equation. Since the argument is of mixed type being retarded and advanced, it offers numerous advantages for the theoretical investigations as well as many real-world applications. The results are confirmed by example and simulation.

Keywords: Unpredictable Oscillations; Poincaré Chaos; Exponential Stability; Piecewise Constant Argument

1. Introduction and Preliminaries

Theory of differential equations provides indispensable tools to study and understand real world processes. Existence of discontinuous characters in nature pushes scientists for developing this theory further. This development enables to obtain more natural features of the real problems modeled by differential equations including discontinuous items. As is well known, the class of differential equations involving discontinuities contains differential equations with piecewise constant argument

(PCA) [3] as a subclass [4]. This subclass has been broadened in [5] by addressing a wider range for PCA, which is entitled as piecewise constant argument of generalized type (PCAG). Although most prior findings for differential equations with PCA were achieved through reduction to discrete equations or numerical approaches [1, 6], equivalent integral equations were employed for the first time in [5] to examine differential equations with PCAG. Thanks to the use of equivalent integral equations, qualitative characteristics of differential equations with PCAG has begun to be examined in a more general manner. This topic has been the focus of various theoretical and practical studies [7–14].

After unpredictable oscillations have been defined as a new type of oscillations in the past few years, a rapid progress has been achieved on the subject [1, 2, 15–21]. This is because, it is seen that they are very useful for the simplification of the chaos analysis for differential equations and also for processes modeled by differential equations. Due to the demands of science and technology, differential equations theory pays great attention to the oscillation phenomena. Accordingly, solutions being periodic, quasi-periodic as well as almost periodic are studied extensively by the researchers [22–24]. It was confirmed for a differential equation that if an unpredictable oscillation exists, then there is Poincaré chaos [1]. As a result of this significant aspect, research on unpredictable solutions is just as beneficial as research on chaos. The main objective of this study is to investigate unpredictable oscillations of a nonlinear differential equation having piecewise alternately retarded and advanced argument of generalized form, which can be considered as the most general case of PCAG since it is of mixed type. Due to the remarkable position of differential equations with PCAG in various applications [9, 10, 25, 26, 27], it is momentous to link these equations with the chaos concept through unpredictable oscillations.

We shall use the abbreviations R, Z and N to represent the set of real numbers, integers and natural numbers, respectively. We pick sequences $\{\chi_j\}_{j\in Z}$ and $\{\eta_j\}_{j\in Z}$ whose elements are real numbers and meet the properties: $\chi_j \le \eta_j \le \chi_{j+1}$ for each $j \in Z$, and $|\chi_j| \to \infty$ as $|j| \to \infty$. The following nonlinear differential equation with piecewise alternately retarded and advanced argument of generalized type is the major focus of the present research:

$$x'(t) = \alpha x(t) + u\big(x(t)\big) + v\Big(x\big(h(t)\big)\Big) + w(t). \qquad (1)$$

Here, t, x belong to the set R, $\alpha < 0$ and $h(t) = \eta_j$ for $t \in [\chi_j, \chi_{j+1})$, $j \in Z$; the functions $u, v:(-a, a) \to R, a > 0,$ are assumed to be continuous. Furthermore, $w: R \to R$ is considered as a bounded and uniformly continuous function.

Definition 1 A bounded, uniformly continuous function $f: R \to R$ is called as unpredictable if one can find two sequences t_{n}, s_{n}, both of them diverging to ∞ and positive constants π_0, ρ such that $f(t + t_{\mathrm{n}}) \to f(t)$ uniformly when $n \to \infty$ on the compact subsets of R and $\pi_0 \le |f(t + t_{\mathrm{n}}) \to f(t)|$ for each $t \in [s_{\mathrm{n}} - \rho, s_{\mathrm{n}} + \rho]$, $n \in N$.

The following criteria are considered to be met during the present investigation.

(H1) $|u(x) - u(y)| \le L_1|x - y|$, $|v(x) - v(y)| \le L_2|x - y|$ for every $x, y \in (-a, a)$ with Lipschitz constants L_u, L_v.

(H2) There exist constants $M_u > 0$, $M_v > 0$ such that $\sup\limits_{x\in(-a,a)} |u(x)| \le M_u$ and $\sup\limits_{x\in(-a,a)} |v(x)| \le M_v$.

(H3) There is a constant $M_w > 0$ so that $\sup\limits_{t\in R}|w(t)| \le M_w$.

(H4) $-\frac{1}{\alpha}(M_u + M_v + M_w) < a$.

(H5) $\frac{1}{\alpha}(L_u + L_v) < 1$.

(H6) For each $\chi_{j+1} - \chi_j \le \chi$, $j \in Z$ for a positive constant χ.

(H7) $\alpha + L_u + \mathcal{H}L_v < 0$, where $\mathcal{H} = \left(1 - \chi(-\alpha + L_u)(1 + L_v\chi)e^{(-\alpha+L_u)\chi} + L_v\right)^{-1}$.

(H8) $\chi\left((-\alpha + L_u)(1 + L_v\chi)e^{(-\alpha+L_u)\chi} + L_v\right) < 1$.

(H9) For a sequence k_n satisfying $k_n \to \infty$ as $n \to \infty$ and for the sequence t_{n} specified in Definition 1, $\chi_{j-k_n} + t_{\mathrm{n}} - \chi_j \to 0$ and $\eta_{j-k_n} + t_{\mathrm{n}} - \eta_j \to 0$ as $n \to \infty$ on each interval, which is finite and comprised of integers.

2. Existence, Uniqueness and Stability of Unpredictable Oscillations

We take a set $\mathcal{D}$ whose elements are real valued functions given by $\Omega: R \to R$ together with the norm defined as $\|\Omega\|_1 = \sup\limits_{t\in R}|\Omega(t)|$. It is adopted that an element Ω of the set $\mathcal{D}$ admits the features listed below:

(D1) Ω is continuous uniformly.

(D2) $\|\Omega\|_1 < a$;

(D3) a sequence t_{n} divergent to ∞ can be found to satisfy the limit $\Omega(t + t_{\mathrm{n}}) \to \Omega(t)$ uniformly on each closed and also bounded interval of real numbers.

Lemma 1 [28]. A necessary and sufficient condition for a bounded function $x(t), t \in R$, to be a solution of (1) is that the integral equation given by

$$x(t) = \int_{-\infty}^{t} e^{\alpha(t-\tau)} \left(u\big(x(\tau)\big) + v\left(x\big(h(\tau)\big)\right) + w(\tau)\right) d\tau, \qquad (2)$$

holds true for $x(t)$.

On the set $\mathcal{D}$, Construct an operator Γ formulated by

$$\Gamma\Omega(t) = \int_{-\infty}^{t} e^{\alpha(t-\tau)} \left(u\big(\Omega(\tau)\big) + v\left(\Omega\big(h(\tau)\big)\right) + w(\tau)\right) d\tau.$$

Theorem 1 Γ is invariant on $\mathcal{D}$.

Proof. We are required to demonstrate that the set $\mathcal{D}$ involves $\Gamma\mathcal{D}$ in it. If we find the derivative of $\Gamma\Omega(t)$ along the independent variable t and then take the absolute value of this derivative, we get

$$\left|\frac{d\Gamma\Omega(t)}{dt}\right| \leq \left|u\big(\Omega(t)\big)\right| + \left|v\left(\Omega\big(h(t)\big)\right)\right| + |w(t)| -$$

$$-\alpha \int_{-\infty}^{t} e^{\alpha(t-\tau)} \left(u\big(\Omega(\tau)\big) + v\left(\Omega\big(h(\tau)\big)\right) + w(\tau)\right) d\tau \leq 2(M_u + M_v + M_w)$$

for each real number t. This means that $\Gamma\Omega$ meets the property (D1), since it is continuous uniformly due to having a bounded derivative.

Note that any element Ω of the set $\mathcal{D}$ satisfies

$$|\Gamma\Omega(t)| \leq -\frac{1}{\alpha}(M_u + M_v + M_w).$$

This implies together with (H4) that (D2) is true for $\Gamma\Omega$.

Now, we are only left to consider (D3). For this purpose, we pick $\delta > 0$ and consider the interval $[\lambda, \mu]$ with $\lambda < \mu$ We can find $\nu < \lambda$ and $\sigma > 0$ in order to fulfill the following four inequalities:

$$-\frac{2}{\alpha}(L_u a + L_v a + M_w)e^{\alpha(\lambda-\nu)} < \frac{\delta}{4}, \qquad (3)$$

$$-\frac{\sigma}{\alpha}(1 + L_u) < \frac{\delta}{4}. \qquad (4)$$

It is true for sufficiently large values of n that $|\Omega(t+t_{\mathrm{n}})-\Omega(t)|<\sigma$ and $|w(t+t_{\mathrm{n}})-w(t)|<\sigma$ whenever $t\in[\nu,\mu]$ and $\left|\chi_{j-k_n}+t_{\mathrm{n}}-\chi_j\right|<\sigma$ whenever $\chi_j\in[\nu,\mu]$, $j\in Z$. Therefore, we have

$$
\begin{aligned}
|\Gamma\Omega(t+t_{\mathrm{n}})-\Gamma\Omega(t)| \le \int_{-\infty}^{c} e^{\alpha(t-\tau)}(L_u|\Omega(\tau+t_{\mathrm{n}})-\Omega(\tau)|+\\
+L_v\left|\Omega\big(h(\tau+t_{\mathrm{n}})\big)-\Omega\big(h(\tau)\big)\right|+|w(t+t_{\mathrm{n}})-w(t)|\big)d\tau+\\
\int_{c}^{t} e^{\alpha(t-\tau)}(L_u|\Omega(\tau+t_{\mathrm{n}})-\Omega(\tau)|+\\
+L_v\left|\Omega\big(h(\tau+t_{\mathrm{n}})\big)-\Omega\big(h(\tau)\big)\right|+|w(t+t_{\mathrm{n}})-w(t)|\big)d\tau\\
\le -\frac{2}{\alpha}(L_u a+L_v a+M_w)e^{\alpha(\lambda-\nu)}-\frac{1}{\alpha}(1+L_u)\sigma+\\
+L_v\int_{c}^{t} e^{\alpha(t-\tau)}\left|\Omega\big(h(\tau+t_{\mathrm{n}})\big)-\Omega\big(h(\tau)\big)\right|d\tau.
\end{aligned}
$$

For $t\in[\lambda,\mu]$ fixed, we assume that $\chi_j\le\chi_{j-k_n}+t_{\mathrm{n}}$ and $\chi_j\le\chi_{j-k_n}+t_{\mathrm{n}}=c<\chi_{j+1}<\chi_{j+2}<\cdots<\chi_{j+r}\le\chi_{j+r-k_n}+t_{\mathrm{n}}\le t<\chi_{j+r+1}$, which does not lose the generality. Thuswise, there exist only r discontinuity moments on $[\nu,t]$: $\chi_{j+1},\chi_{j+2},\dots,\chi_{j+r}$. Moreover, assume that

$$-\frac{2(r+1)L_v\,\sigma}{\alpha}(1-e^{\alpha\chi})<\frac{\delta}{4}, \tag{5}$$

$$-\frac{2rL_v\,a}{\alpha}(e^{-\alpha\sigma}-1)<\frac{\delta}{4}. \tag{6}$$

Next, we shall focus on the integral

$$\int_{c}^{t} e^{\alpha(t-\tau)}\left|\Omega\big(h(\tau+t_{\mathrm{n}})\big)-\Omega\big(h(\tau)\big)\right|d\tau.$$

In fact, this integral can be written as follows

$$\sum_{j=k}^{k+r-1} \int_{\chi_{j-k_n}+t_n}^{\chi_{j+1}} e^{\alpha(t-\tau)} \left|\Omega\big(h(\tau+t_n)\big) - \Omega\big(h(\tau)\big)\right| d\tau +$$

$$+ \sum_{j=k}^{k+r-1} \int_{\chi_{j+1}}^{\chi_{j+1-k_n}+t_n} e^{\alpha(t-\tau)} \left|\Omega\big(h(\tau+t_n)\big) - \Omega\big(h(\tau)\big)\right| d\tau +$$

$$+ \int_{\chi_{k+r-k_n}+t_n}^{t} e^{\alpha(t-\tau)} \left|\Omega\big(h(\tau+t_n)\big) - \Omega\big(h(\tau)\big)\right| d\tau.$$

If $\chi_{i-k_n} + t_n \le t < \chi_{i+1}$ for some $i \in Z$, then it can be seen that $h(t) = \eta_i$, and in turn it is true due to condition (H9) that $h(t+t_n) = \eta_{i+k_n}$. Thus, we attain that

$$\int_{\chi_{i-k_n}+t_n}^{\chi_{i+1}} e^{\alpha(t-\tau)} \left|\Omega\big(\eta_{i+k_n}\big) - \Omega(\eta_i)\right| d\tau =$$

$$= \int_{\chi_{i-k_n}+t_n}^{\chi_{i+1}} e^{\alpha(t-\tau)} |\Omega(\eta_i + t_n) - \Omega(\eta_i) + \Omega(\eta_i + t_n + o(1)) - \Omega(\eta_i + t_n)| \, d\tau$$

$$\le \int_{\chi_{i-k_n}+t_n}^{\chi_{i+1}} e^{\alpha(t-\tau)} \Big|\sigma + \left|\Omega\big(\eta_i + t_n + o(1)\big) - \Omega(\eta_i + t_n)\right|\Big| \, d\tau.$$

Recall that Ω is continuous uniformly. As a result, given $\sigma > 0$ and for big enough n, there is a $\kappa > 0$ so that $\left|\Omega\big(\eta_i + t_n + o(1)\big) - \Omega(\eta_i + t_n)\right| < \sigma$ as long as $\left|\eta_{i-k_n} + t_n - \eta_i\right| < \kappa$. After all, we conclude that

$$\sum_{j=k}^{k+r-1} \int_{\chi_{j-k_n}+t_n}^{\chi_{j+1}} e^{\alpha(t-\tau)} \left|\Omega\big(h(\tau+t_n)\big) - \Omega\big(h(\tau)\big)\right| d\tau \le 2p\sigma \int_{\chi_{i-k_n}+t_n}^{\chi_{i+1}} e^{\alpha(t-\tau)} \, d\tau \le -\frac{2p\sigma}{\alpha}(1 - e^{\alpha\chi})$$

for the integer values of i that lies in the range $l \le i \le l + r - 1$. Analogously, it can be achieved that

$$\int_{\chi_{i+r-1-k_n}+t_n}^{t} e^{\alpha(t-\tau)} \left|\Omega\big(\eta_{i+k_n}\big) - \Omega(\eta_i)\right| d\tau = -\frac{2\sigma}{\alpha}(1 - e^{\alpha\chi}).$$

Likewise, owing to the assumption given by (H9), we find that

$$\sum_{j=k}^{k+r-1} \int_{\chi_{j+1}}^{\chi_{j+1-k_n}+t_{\mathrm{n}}} e^{\alpha(t-\tau)} \left|\Omega\big(h(\tau+t_{\mathrm{n}})\big)-\Omega\big(h(\tau)\big)\right| d\tau \le 2ra \int_{\chi_{i+1}}^{\chi_{i+1-k_n}+t_{\mathrm{n}}} e^{\alpha(t-\tau)}\, d\tau$$

$$= -\frac{2ra}{\alpha}\left(e^{-\alpha\sigma}-1\right)$$

for the integer values of i lying in the range $k \le i \le k+r-1$. As a result of these computations performed above, we get

$$\int_{c}^{t} e^{\alpha(t-\tau)} \left|\Omega\big(h(\tau+t_{\mathrm{n}})\big)-\Omega\big(h(\tau)\big)\right| d\tau \le -\frac{2(r+1)\sigma}{\alpha}\left(1-e^{\alpha\chi}\right)-\frac{2ra}{\alpha}\left(e^{-\alpha\sigma}-1\right).$$

Hence, the following inequality

$$|\Gamma\Omega(t+t_{\mathrm{n}})-\Gamma\Omega(t)| \le -\frac{2}{\alpha}\left(L_u a+L_v a+M_w\right)e^{\alpha(\lambda-\nu)}-\frac{\sigma}{\alpha}\left(1+L_u\right)+$$

$$-\frac{2(r+1)\sigma}{\alpha}\left(1-e^{\alpha\chi}\right)-\frac{2ra}{\alpha}\left(e^{-\alpha\sigma}-1\right)$$

is valid for all $t \in [\lambda,\mu]$.

Using the inequalities (3)-(6), we obtain for each $t \in [\lambda,\mu]$ that $|\Gamma\Omega(t+t_{\mathrm{n}})-\Gamma\Omega(t)| < \delta$. This shows that (D3) is fulfilled by Γ as well. The proof is completed.

It is not difficult to prove by assumption (H5) that Γ is a contractive operator on $\mathcal{D}$. Hence, we give this result below and omit the proof here.

Theorem 2 Γ is a contractive operator on $\mathcal{D}$.

Next, we state an auxiliary result which can be proven by using a classical theory of differential equations and condition (H8) [2,4].

Lemma 2. Assume that the conditions (H1), (H6) and (H8) are fulfilled. If $\psi(t)$ is a solution of

$$\psi'(t) = \alpha\psi(t) + u\big(\psi(t) + \varphi(t)\big)u - \big(\psi(\varphi(t)\big) +$$
$$+v\Big(\psi\big(h(t)\big) + \varphi\big(h(t)\big)\Big) - v\Big(\varphi\big(h(t)\big)\Big), \qquad (7)$$

where $\varphi(t)$ is a continuous function with $\|\varphi(t)\|_1 < a$, then $\big|\psi\big(h(t)\big)\big| \leq \mathcal{H}$ is satisfied for all $t \in (-\infty, \infty)$.

Now, we are ready to state the main result of the present paper.

Theorem 3. Let all conditions from (H1) to (H9) hold true and the function w be unpredictable. Then, equation (1) possesses a unique exponentially stable unpredictable oscillation.

Proof. It follows from the results in [2, 28] that $\mathcal{D}$ is complete. Hence, the operator Γ has a unique fixed point $\varphi(t)$ which belongs to the set $\mathcal{D}$. Actually, $\varphi(t)$ is the unique solution of (1).

Next, we will show that the function $\varphi(t)$ is unpredictable. m_1, m_2 and ϵ are chosen to satisfy

$$\epsilon < \rho, \qquad (8)$$

$$\epsilon\left((\alpha - L_u)\left(\frac{1}{m_1} + \frac{2}{m_2}\right) - 2L_v + \frac{1}{2}\right) < \frac{3}{2m_1}, \qquad (9)$$

$$|\varphi(t+\tau) - \varphi(t)| < \pi_0 min\left\{\frac{1}{4m_1}, \frac{1}{m_2}\right\}, t \in R, |\tau| < \epsilon. \qquad (10)$$

We fix ϵ, m_1, m_2 and $n \in N$ and define $\Pi = |\varphi(s_n + t_n) - \varphi(s_n)|$.

Case I: Let $\Pi \geq 0$. Then, we obtain for $t \in [s_n - \epsilon, s_n + \epsilon]$, $n \in N$, that

$$|\varphi(t + t_n) - \varphi(t)| \geq |\varphi(s_n + t_n) - \varphi(s_n)| - |\varphi(s_n) - \varphi(t)| -$$
$$-|\varphi(t + t_n) - \varphi(s_n + t_n)| > \frac{\pi_0}{m_1} - \frac{\pi_0}{4m_1} - \frac{\pi_0}{4m_1} = \frac{1}{2m_1}\pi_0.$$

Case II: Let $\Pi < 0$. Then, the inequality (10) implies for $t \in [s_n, s_n + \epsilon]$, $n \in N$ that

$$|\varphi(t + t_n) - \varphi(t)| \leq |\varphi(s_n + t_n) - \varphi(s_n)| + |\varphi(s_n) - \varphi(t)| +$$
$$+|\varphi(t + t_n) - \varphi(s_n + t_n)| < \frac{\pi_0}{m_1} + \frac{\pi_0}{m_2} + \frac{\pi_0}{m_2} = \left(\frac{1}{m_1} + \frac{2}{m_2}\right)\pi_0.$$

Since the integral equation given by

$$\varphi(t) = \varphi(s_{\mathrm{n}}) + \int_{s_{\mathrm{n}}}^{t} \Big(\alpha\varphi(\tau) + u\big(\varphi(\tau)\big) + v\Big(\varphi\big(h(\tau)\big)\Big) + w(\tau)\Big)\, d\tau.$$

is valid, we can conlude for $t \in \left[s_{\mathrm{n}} + \frac{\epsilon}{2}, s_{\mathrm{n}} + \epsilon\right]$, $n \in N$, that

$$|\varphi(t + t_{\mathrm{n}}) - \varphi(t)| = |\varphi(s_{\mathrm{n}} + t_{\mathrm{n}}) - \varphi(s_{\mathrm{n}}) + a \int_{s_{\mathrm{n}}}^{t} \big(\varphi(\tau + t_{\mathrm{n}}) - \varphi(\tau)\big) d\tau +$$

$$+ \int_{s_{\mathrm{n}}}^{t} \Big(u\big(\varphi(\tau + t_{\mathrm{n}})\big) - \big(\varphi(\tau)\big)\Big)\, d\tau +$$

$$+ \int_{s_{\mathrm{n}}}^{t} \Big(v\Big(\varphi\big(h(\tau + t_{\mathrm{n}})\big)\Big) - v\Big(\varphi\big(h(\tau)\big)\Big)\Big)\, d\tau + \int_{s_{\mathrm{n}}}^{t} \big(w(\tau + t_{\mathrm{n}}) - w(\tau)\big) d\tau\Big| \geq$$

$$-\frac{\pi_0}{m_1} + \alpha\epsilon\left(\frac{1}{m_1} + \frac{2}{m_2}\right)\pi_0 - L_u\epsilon\left(\frac{1}{m_1} + \frac{2}{m_2}\right)\pi_0 -$$

$$-L_v \int_{s_{\mathrm{n}}}^{t} \Big|\varphi\big(h(\tau + t_{\mathrm{n}})\big) - \varphi\big(h(\tau)\big)\Big| d\tau + \frac{\epsilon}{2}\pi_0.$$

Next, we fix t from the interval $t \in \left[s_{\mathrm{n}} + \frac{\epsilon}{2}, s_{\mathrm{n}} + \frac{\epsilon}{2}\right]$ and pick ϵ sufficiently small in order to satisfy $\chi_{j-k_n} + t_{\mathrm{n}} \leq s_{\mathrm{n}} < s_{\mathrm{n}} + \frac{\epsilon}{2} \leq t \leq s_{\mathrm{n}} + \epsilon < \chi_{j+1}$ for some integer j. For $t \in \left[s_{\mathrm{n}} + \frac{\epsilon}{2}, s_{\mathrm{n}} + \epsilon\right]$, it is true that $h(t) = \eta_i$ and hence $h(t + t_{\mathrm{n}}) = \eta_{i+k_n}$ because of the assumption (H9). Since the function $\varphi(t) \in \mathcal{D}$ is continuous uniformly, given $\pi_0 > 0$ and for big enough n a number $\beta > 0$ can be found satisfying $\left|\varphi\big(\eta_{i+k_n}\big) - \varphi(\eta_i)\right| \leq 2\pi_0$ as long as $\left|\eta_{i+k_n} - \eta_i - t_{\mathrm{n}}\right| < \beta$. Therefore, it follows that

$$\int_{s_{\mathrm{n}}}^{t} \Big|\varphi\big(h(\tau + t_{\mathrm{n}})\big) - \varphi\big(h(\tau)\big)\Big| d\tau \leq 2\pi_0.$$

If we use inequality (9), then we see that

$$|\varphi(t+t_{\mathrm{n}})-\varphi(t)| \geq -\frac{\pi_0}{m_1}+\alpha\left(\frac{1}{m_1}+\frac{2}{m_2}\right)\epsilon\pi_0 - L_u\left(\frac{1}{m_1}+\frac{2}{m_2}\right)\epsilon\pi_0 -$$

$$-2L_v\epsilon\pi_0+\frac{\epsilon}{2}\pi_0 \geq -\frac{\pi_0}{m_1}+\frac{3\pi_0}{2m_1} \geq \frac{\pi_0}{2m_1},$$

which means that $\varphi(t)$ is an unpredictable oscillation according to the Definition 1.

Lastly, we are left to show that this unpredictable oscillation is exponentially stable. Let $y(t)$ be another solution of (1). Construct the difference $\psi(t)=y(t)-\varphi(t)$. It can be shown easily that $\psi(t)$ is a solution of (7). Hence, we have

$$|\psi(t)| \leq e^{\alpha(t-t_0)}|\psi(t_0)| + \int_c^t e^{\alpha(t-t_0)}\left(L_u|\psi(\tau)| + L_v\left|\psi\left(h(\tau)\right)\right|\right)d\tau.$$

Rearranging the last inequality and applying the Gronwall-Bellman Lemma, we find that

$$|y(t)-\varphi(t)| \leq e^{\alpha(t-t_0)}|y(t_0)-\varphi(t_0)| + e^{(\alpha+L_u+L_v\mathcal{H})(t-t_0)}.$$

which gives together with (H7) that the solution $\varphi(t)$ is stable exponentially. The proof is completed.

3. Example and Simulation

We give example with numerical simulations to illustrate the theoretical results. To investigate the presence of an unpredictable solution, we need to use the logistic map $z_{i+1}=cz_i(1-z_i), i \in \mathbb{Z}$. According to the Theorem 4.1 [29], for each $c \in [3+(2/3)^{1/2}, 4]$, the logistic map possesses an unpredictable solution. Let z_i, $t \in [\chi_i,\ \chi_{i+1}), i \in \mathbb{Z}$ be an unpredictable solution of the logistic map with $\mu = 3.92$.

As a perturbation, we will use an unpredictable function $\Phi(t)$ [30]:

$$\Phi(t) = \int_{-\infty}^{t} e^{-4(t-\tau)}\Omega(\tau)d\tau\,, t \in R,$$

where $\Omega(t) = z_i$ for $t \in [\chi_i,\ \chi_{i+1}), i \in \mathbb{Z}$. It is worth noting that $\Phi(t)$ is bounded on the whole real axis such that $\sup_{t\in\mathbb{R}}|\Phi(t)| \leq 1/4$.

Consider the following scalar nonlinear differential equation with piecewise constant argument of generalized type,

$$x'(t) = -0.5x(\mathrm{t}) + 0.04\tanh\left(\frac{x(\mathrm{t})}{4}\right) + 0.04\tanh\left(\frac{x(\mathrm{h(t)})}{5}\right) - 18\Phi^3(t) + 1.4, \quad (11)$$

where the argument function $h(t) = \eta_k$ is defined by the sequences $\chi_k = k, \quad \eta_k = \frac{\chi_k + \chi_{k+1}}{2} + z_k = \frac{2k+1}{2} + z_k, \ k \in Z$. Moreover, $w(\mathrm{t}) = -18\Phi^3(t) + 1.4$ is unpredictable functions in accordance with Lemmas 1.4 and 1.5 given in [15].

We can see that the conditions $(H1)$ -(H9) are valid for the system (11) with $L_u = 0{,}01, L_v = 0.008$, $M_u = M_v = 0.04$ and $M_w = 1.68$, $a = 4$. Thus, by the Theorem 3, system (11) has a unique exponentially stable unpredictable solution.

This is typically an unsolvable problem of finding the initial value of the solution, since it is impossible. According to the property of exponential stability, any solution from the domain approaches to the unpredictable solution. That is, to imagine the behavior of the unpredictable solution $x(t)$, we consider the simulation of another solution $\theta(t)$, with initial values $\theta(0) = 1.859$ in Fig 1.

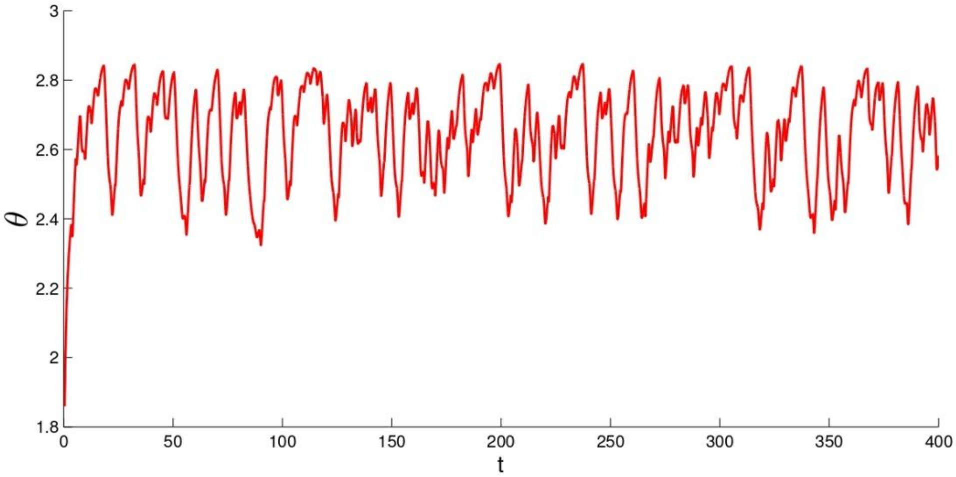

Fig. 1 Graph of the function $\theta(t)$.

ACKNOWLEDGEMENTS

M. Tleubergenova and Z. Nugayeva are supported by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grants No. AP09258737 and No. AP08856170). M. Akhmet is supported by 2247-A National Leading Researchers Program of TUBITAK, Turkey, N 120C138.

References

1. Akhmet, M., Fen, M.O. Existence of unpredictable solutions and chaos // Turk. J. Math. – 2017. – Vol. 41, – P. 254-266.

2. Akhmet, M., Aruğaslan Çinçin, D., Tleubergenova, M., Nugayeva, Z. Unpredictable oscillations for Hopfield-type neural networks with delayed and advanced arguments // Mathematics. –2021. – Vol. 9, №571. – P. 1-19.

3. Wiener, J. Generalized Solutions of Functional Differential Equations // World Scientific – Singapore, 1993.

4. Akhmet, M. Nonlinear Hybrid Continuous/Discrete-Time Models. // Atlantis Press – Paris, 2011.

5. Akhmet, M.U. Integral manifolds of differential equations with piecewise constant argument of generalized type // Nonlinear Analysis – 2007. –Vol. 66, – P. 367–383.

6. Cooke, K.L.,Wiener, J. Retarded differential equations with piecewise constant delays // J. Math. Anal. Appl. – 1984. –Vol. 99, – P. 265–297.

7. Akhmet, M.U. (2008). Stability of differential equations with piecewise constant arguments of generalized type // Nonlinear Analysis – 2008. –Vol. 68, –P. 794–803.

8. Akhmet, M.U., Aruğaslan, D. (2009). Lyapunov-Razumikhin method for differential equations with piecewise constant argument // Discrete and Contiunous Dynamical Systems-Series A. – 2009. –Vol. 25, – P. 457–466.

9. Akhmet, M.U., Aruğaslan, D., Yılmaz, E. Stability analysis of recurrent neural networks with piecewise constant argument of generalized type // Neural Networks. – 2010. –Vol. 23, – P. 805–811.

10. Akhmet, M.U., Aruğaslan, D., Yılmaz, E. Stability in cellular neural networks with a piecewise constant argument // J. Comput. Appl. Math. – 2010. –Vol. 233, – P. 2365–2373.

11. Akhmet, M.U., Aruğaslan, D., Cengiz, N. Exponential stability of periodic solutions of recurrent neural networks with functional dependence on piecewise constant argument // Turk. J. Math. – 2018. –Vol. 42, – P. 272–292.

12. Aruğaslan Çinçin, D., Cengiz, N. Qualitative behavior of a Lienard-type differential equation with piecewise constant delays // Iran. J. Sci. Technol. Trans. Sci. – 2020. –Vol. 44, – P. 1439–1446.

13. Wu, A., Liu, L., Huang, T., Zeng, Z. Mittag-Leffler stability of fractional-order neural networks in the presence of generalized piecewise constant arguments // Neural Networks – 2017. – Vol. 85, – P. 118–127.

14. Xi, Q. Razumikhin-type theorems for impulsive differential equations with piecewise constant argument of generalized type // Adv. Differ. Equ. – 2018. –Vol. 267, – P. 1–16.

15. Akhmet, M., Fen, M.O., Tleubergenova, M., Zhamanshin, A. Unpredictable solutions of linear differential and discrete equations // Turk. J. Math. – 2019. –Vol. 43, – P. 2377–2389.

16. Akhmet, M., Tleubergenova, M., Zhamanshin, A. Quasilinear differential equations with strongly unpredictable solutions // Carpathian J. Math. – 2020. –Vol. 36, – P. 341–349.

17. Akhmet, M., Tleubergenova, M., Fen, M.O., Nugayeva, Z. Unpredictable solutions of linear impulsive systems // Mathematics. – 2020. –Vol. 8, №1798.

18. Akhmet, M., Tleubergenova, M., Nugayeva, Z. (2020). Strongly unpredictable oscillations of Hopfield-type neural networks // Mathematics. – 2020. –Vol. 8, №1791.

19. Akhmet, M., Seilova, R., Tleubergenova, M., Zhamanshin, A. Shunting inhibitory cellular neural networks with strongly unpredictable oscillations // Commun. Nonlinear Sci. Numer. Simulat. – 2020. –Vol. 89, №105287.

20. Akhmet, M., Tleubergenova, M., Zhamanshin, A. Inertial neural networks with unpredictable oscillationsr // Mathematics. – 2020. –Vol. 8, №1797.

21. Akhmet, M., Fen, M.O., Tleubergenova, M., Zhamanshin, A. Poincare chaos for a hyperbolic quasilinear system // Miskolc Mathematical Notes. – 2019. –Vol. 20, – P. 33–44.

22. Farkas, M. Periodic Motion // Springer – New York, 1994.

23. Corduneanu, C. Almost Periodic Oscillations and Waves // Springer – New York, 2009.

24. Akhmet, M.U. Almost Periodicity, Chaos, and Asymptotic Equivalence. Springer –New York, 2020.

25. Pinto, M., Sepulveda, D., Torres, R. Exponential periodic attractor of impulsive Hopfield-type neural network system with piecewise constant argument // Electron. J. Qual. Theory Differ. Equ. – 2018. –Vol. 34, – P. 1–28.

26. Aruğaslan, D., Cengiz, N. Existence of periodic solutions for a mechanical system with piecewise constant forces // Hacet. J. Math. Stat. – 2018. –Vol. 47, – P. 521–538.

27. Aruğaslan, D., Özer, A. Stability analysis of a predator-prey model with piecewise constant argument of generalized type using Lyapunov functions //
J. Math. Sci. (N. Y.) – 2014. –Vol. 203, – P. 297–305.

28. Hartman, P. Ordinary Differential Equations. Birkhauser – Boston, 1982.

29. Akhmet M., Fen M.O. Poincare chaos and unpredictable functions // Communications in Nonlinear Science and Numerical Simulation. – 2017. – № 48. - P. 85-94.

30. Akhmet M., Fen M.O. Non-autonomous equations with unpredictable solutions // Communications in Nonlinear Sciences and Numerical Simulation. – 2018. – №59.– P. 657-670.

СЫЗЫҚТЫҚ ЕМЕС ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ЖАЛПЫЛАНҒАН ТИПТІ КЕЗЕКПЕН КЕШІКТІРІЛГЕН ЖӘНЕ ІЛГЕРІЛЕГЕН БӨЛІКТІ АРГУМЕНТТІ БОЛЖАНБАЙТЫН ТЕРБЕЛІСТЕРІНІҢ БАР БОЛУЫ ЖӘНЕ ОРНЫҚТЫЛЫҒЫ ТУРАЛЫ

Д. АРУГАСЛАН ЧИНЧИН[1], М. ТЛЕУБЕРГЕНОВА[2,3], З. НУГАЕВА[2,3], М. АХМЕТ[4]

[1] *Сулейман Демирель Университеті, Испарта, Түркия*

[2] *Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан*

[3] *Ақпараттық және есептеуіш технологиялар институты, Алматы, Қазақстан*

[4] *Орталық-Шығыс Техникалық Университеті, Анкара, Түркия*

E-mail: duyguarugaslan@sdu.edu.tr; madina_1970@mail.ru; zahira2009.85@mail.ru; marat@metu.edu.tr

Аңдатпа. Соңғы зерттеулер дифференциалдық теңдеу үшін болжанбайтын тербелістің бар болуы Пуанкаре хаосының бар болуымен бірдей екенін көрсетті [1,2]. Осыған байланысты, бұл жұмыста жалпыланған типтегі кезекпен кешіктірілген және ілгерілеген бөлікті аргументі сызықтық емес дифференциалдық теңдеу үшін болжанбайтын тербелістерді талдау негізі мақсат болып табылады. Қарастырылып отырған теңдеу үшін болжанбайтын тербелістердің бар болуы-жалғыздығы мен экспоненциалды орнықтылығы тексеріледі. Аргумент аралас, яғни, кешіктірілген және ілгерілеген түрге ие болғандықтан, ол теориялық зерттеулер үшін де, нақты қосымшалар үшін де көптеген артықшылықтарды ұсынады. Нәтижелер мысалдар мен модельдеу арқылы расталады.

Түйін сөздер: болжанбайтын тербелістер; Пуанкаре хаосы; экспоненциалды орнықтылық; бөлікті-тұрақты аргумент.

О СУЩЕСТВОВАНИИ И УСТОЙЧИВОСТИ НЕПРЕДСКАЗУЕМЫХ КОЛЕБАНИЙ ДЛЯ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С КУСОЧНО-ПООЧЕРЕДНО ЗАПАЗДЫВАЮЩИМ И ОПЕРЕЖАЮЩИМ АРГУМЕНТАМ ОБОБЩЕННОГО ТИПА

Д. АРУГАСЛАН ЧИНЧИН[1], М. ТЛЕУБЕРГЕНОВА[2,3], З. НУГАЕВА[2,3], М. АХМЕТ[4]

[1] *Университет Сулеймана Демиреля, Испарта, Турция*

[2] *Актюбинский региональный университет им. К.Жубанова, Актобе, Казахстан*

[3] *Институт информационных и вычислительных технологий, Алматы, Казахстан*

[4] *Средневосточный технический университет, Анкара, Турция*

E-mail: duyguarugaslan@sdu.edu.tr; madina_1970@mail.ru; zahira2009.85@mail.ru; marat@metu.edu.tr

Аннотация. Недавние исследования показали, что наличие непредсказуемого колебания для дифференциального уравнения равносильно наличию хаоса Пуанкаре [1,2]. В связи с этим в данной работе ставится цель проанализировать непредсказуемые колебания для нелинейного дифференциального уравнения с кусочно-поочередно запаздывающим и опережающим аргументом обобщенного типа. Для рассматриваемого уравнения проверяются существование-единственность и экспоненциальная устойчивость непредсказуемых колебаний. Поскольку аргумент имеет смешанный тип, запаздывающий и опережающий, он предлагает многочисленные преимущества для теоретических исследований, а также для многих реальных приложений. Результаты подтверждены примером и моделированием.

Ключевые слова: непредсказуемые колебания; хаос Пуанкаре; экспоненциальная устойчивость; кусочно-постоянный аргумент.

UDC 517.52